Preprint typeset in JHEP style. - PAPER VERSION

DPNU-02-22
IU-MSTP/51
hep-lat/0207009



# Chiral anomalies in the reduced model

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ABSTRACT: On the basis of an observation due to Kiskis, Narayanan and Neuberger, we show that there is a remnant of chiral anomalies in the reduced model when a Dirac operator which obeys the Ginsparg-Wilson relation is employed for the fermion sector. We consider fermions belonging to the fundamental representation of the gauge group $\mathrm{U}(N)$ or $\mathrm{SU}(N)$. For vector-like theories, we determine a general form of the axial anomaly or the topological charge within a framework of a $\mathrm{U}(1)$ embedding. For chiral gauge theories with the gauge group $\mathrm{U}(N)$, a remnant of gauge anomaly emerges as an obstruction to a smooth fermion integration measure. The pure gauge action of gauge-field configurations which cause these non-trivial phenomena always diverges in the 't Hooft $N \to \infty$ limit when $d > 2$.

KEYWORDS: Renormalization Regularization and Renormalons, Lattice Gauge Field Theories, Gauge Symmetry, Anomalies in Field and String Theories.

## 1. Introduction

In a recent paper [1], Kiskis, Narayanan and Neuberger proposed a use of the overlap-Dirac operator [2] in the quenched reduced model for the large $N$ QCD [3]–[9] (for a more complete list of references, see ref. [10]).[1] In particular, they pointed out that it is possible to define a topological charge $Q$ in the reduced model in the spirit of the overlap [13, 14]. Using the abelian background of ref. [15], they explicitly demonstrated that certain configurations in the reduced model lead to $Q \neq 0$ for $d = 2$ and $d = 4$. They also argued that there may exist some remnant of the gauge anomaly in reduced chiral gauge theories. These observations show an interesting possibility that phenomena related to chiral anomalies in the continuum gauge theory emerge even in the reduced model, although one would naively expect there is no counterpart of chiral anomalies in the reduced model in which spatial dependences of the gauge field are "reduced".

In this paper, we investigate this possibility further with a use of the overlap- or a more general Dirac operator which obeys the Ginsparg-Wilson relation [16, 17]. For our study, an exact correspondence between the reduced model with restricted configurations and a U(1) gauge theory defined on a finite-size lattice will be a basic tool. We thus first clarify how to "embed" a U(1) lattice gauge theory in the reduced model when fermion fields are belonging to the fundamental representation of U($N$) or SU($N$) (section 2). Next, in section 3, after characterizing the above topological charge $Q$ as the axial anomaly in the reduced model, we determine its general form within the U(1) embedding. For this, a knowledge on the axial anomaly on finite-size lattices [18] is crucial; this knowledge is obtained by combining cohomological analyses on the axial anomaly [19]–[24], a complete classification of "admissible" U(1) gauge configurations [25] and the locality of the Dirac operator [26, 27]. We also show that, within the U(1) embedding, the pure gauge action of any configuration with $Q \neq 0$ diverges in the 't Hooft $N \to \infty$ limit; only exception is $d = 2$. In section 4, we study reduced chiral gauge theories along the line of refs. [25, 28] and show that there exists an obstruction to a smooth fermion integration measure over the space of admissible reduced gauge fields; this obstruction might be regarded as a remnant of the gauge anomaly. To show the obstruction, we utilize Lüscher's topological field in $d+2$-dimensional space [28] and the cohomological analysis applied to it [22]. Finally, in section 5, we give a list of open questions and suggest directions of further study.

## 2. U(1) embedding

In the most part of this paper, we focus only on the fermion sector and the gauge field is treated as a non-dynamical background. In the reduced model, the fermion

[1] A similar proposal has been made [11] in the context of the IIB matrix model [12].

action would be read as

$$S_F = \overline{\psi} D \psi, \tag{2.1}$$

where $\psi$ and $\overline{\psi}$ are constant Grassman variables belonging to the fundamental representation of U($N$) or SU($N$). The Dirac operator $D$ defines a coupling of the fermion to the reduced gauge field $U_\mu$. In the case of the quenched reduced model [4, 5, 6], the Dirac operator should be defined with a momentum insertion by the factor $e^{ip_\mu}$. As we will see below, such a global phase factor can be absorbed into the U(1) gauge field within the U(1) embedding. So we will omit the momentum factor in the following discussion.

The basic idea of an "embedding" is to identify the index $n$ ($1 \leq n \leq N$) of the fundamental representation with the coordinate $x$ on a lattice with the size $L$; $\Gamma = \{ x \in \mathbb{Z}^d \mid 0 \leq x_\mu < L \}$. We set $N = L^d$ and adopt the convention between these two:

$$n(x) = 1 + x_d + L x_{d-1} + \cdots + L^{d-1} x_1, \tag{2.2}$$

where $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$. Note that $1 \leq n(x) \leq L^d = N$. With this mapping, a row vector $f_n$ is regarded as a function on the lattice $f(x)$; $f(x) = f_{n(x)}$. The shift operation on the lattice[2]

$$T^0_\mu f(x) = f(\widetilde{x + \hat{\mu}}), \tag{2.3}$$

where $\widetilde{x}_\mu = x_\mu \bmod L$, is then expressed by an action of the $N \times N$ matrix

$$T^0_\mu = 1 \otimes \cdots \otimes 1 \otimes X \otimes 1 \otimes \cdots 1, \tag{2.4}$$

where the factor $X$ appears in the $\mu$-th slot and each elements of the tensor product are $L \times L$ matrices. The unitary matrix $X$ is given by

$$X = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} = V S V^\dagger, \tag{2.5}$$

and

$$S = \begin{pmatrix} 1 & & & & \\ & \eta & & & \\ & & \eta^2 & & \\ & & & \ddots & \\ & & & & \eta^{(L-1)} \end{pmatrix}, \qquad \eta = e^{2\pi i/L}, \tag{2.6}$$

because $X^L = 1$.[3] In fact, one verifies

$$(T^0_\mu f)_{n(x)} = f_{n(\widetilde{x+\hat{\mu}})} = f(\widetilde{x + \hat{\mu}}). \tag{2.7}$$

[2] $\hat{\mu}$ denotes the unit vector in direction $\mu$.

[3] Thus $\det T^0_\mu = (\det S)^{L^{d-1}} = e^{\pi i L^{d-1}(L-1)} = 1$ for $d > 1$.

We may also define a *diagonal* $N \times N$ matrix from a function $f(x)$ by

$$f_{m(x)n(y)} = f_{n(x)}\delta_{m(x)n(y)} = f(x)\delta_{m(x)n(y)}. \tag{2.8}$$

On this matrix, the shift is expressed by the conjugation

$$(T^0_\mu f T^{0\dagger}_\mu)_{m(x)n(y)} = f_{m(\widetilde{x+\hat{\mu}})n(\widetilde{y+\hat{\mu}})} = f(\widetilde{x+\hat{\mu}})\delta_{m(x)n(y)}. \tag{2.9}$$

Now, the gauge coupling in the Dirac operator is always defined through the covariant derivative. For the reduced model, the covariant derivative would be read as

$$\nabla_\mu \psi = U_\mu \psi - \psi. \tag{2.10}$$

We *assume* that the reduced gauge field $U_\mu$ has the following form

$$U_\mu = u_\mu T^0_\mu, \tag{2.11}$$

with a *diagonal* matrix

$$(u_\mu)_{m(x)n(y)} = (u_\mu)_{m(x)}\delta_{m(x)n(y)} = u_\mu(x)\delta_{m(x)n(y)}. \tag{2.12}$$

Since $u_\mu$ is a unitary matrix,[4] the diagonal elements are pure phase, $(u_\mu)_{m(x)} = u_\mu(x) \in \mathrm{U}(1)$. We recall that in the conventional lattice gauge theory the gauge coupling is defined through

$$\begin{aligned} \nabla_\mu \psi(x) &= U_\mu(x)\psi(x+\hat{\mu}) - \psi(x) \\ &= U_\mu(x) T^0_\mu \psi(x) - \psi(x). \end{aligned} \tag{2.13}$$

Comparing this with eqs. (2.10) and (2.11), we realize that when the gauge field in the reduced model $U_\mu$ has the particular form (2.11), the fermion sector in the reduced model is completely identical to that of the conventional U(1) gauge theory defined on a lattice with the size $L$ ($N = L^d$). The U(1) link variables in the latter is given by the diagonal elements of the $N \times N$ matrix $u_\mu$. We call eq. (2.11) the U(1) embedding in this sense.

This identification has a gauge covariant meaning. Namely, the assumed form (2.11) is preserved under the gauge transformation in the reduced model

$$U_\mu \to \Omega U_\mu \Omega^\dagger, \tag{2.14}$$

provided that $\Omega \in \mathrm{U}(N)$ or $\Omega \in \mathrm{SU}(N)$ is a *diagonal* matrix. This transformation induces a transformation on $u_\mu$

$$u_\mu \to \Omega u_\mu (T^0_\mu \Omega^\dagger T^{0\dagger}_\mu), \tag{2.15}$$

[4]When the gauge group is SU($N$), we have an additional constraint that $\det u_\mu = 1$ or $\prod_{x\in\Gamma} u_\mu(x) = 1$.

that is nothing but the conventional U(1) gauge transformation due to eq. (2.9).

Also the plaquette variable in the reduced model and that of the U(1) theory have a simple relation under eq. (2.11). We note[5]

$$U_{\mu\nu} = U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger = u_\mu (T_\mu^0 u_\nu T_\mu^{0\dagger})(T_\nu^0 u_\mu^\dagger T_\nu^{0\dagger}) u_\nu^\dagger, \tag{2.16}$$

is a diagonal matrix and the diagonal $(m(x)m(x))$ element of this equation is the U(1) plaquette:

$$\begin{aligned}(U_{\mu\nu})_{m(x)m(x)} &= (u_\mu)_{m(x)m(x)}(u_\nu)_{m(x+\hat\mu)m(x+\hat\mu)}(u_\mu)^*_{m(x+\hat\nu)m(x+\hat\nu)}(u_\nu)^*_{m(x)m(x)}\\ &= u_{\mu\nu}(x),\end{aligned} \tag{2.17}$$

from eq. (2.9).

In the following, we utilize the above equivalence of the U($N$) or SU($N$) reduced model with restricted configurations and a U(1) gauge theory defined on the finite lattice $\Gamma$. Fortunately, when a Dirac operator which obeys the Ginsparg-Wilson relation is employed, we may invoke a cohomological analysis and related techniques which tell a structure of chiral anomalies on a lattice with finite lattice spacings [19, 20, 21, 22, 23, 24] and with finite sizes [18]. We will fully use these powerful machineries to investigate possible chiral anomalies in the reduced model.

## 3. Axial anomaly and the topological charge

Consider the average over fermion variables in the reduced model

$$\langle \mathcal{O} \rangle_{\mathrm{F}} = \int \mathrm{d}\psi \mathrm{d}\overline{\psi}\, \mathcal{O} \exp(-\overline{\psi} D \psi), \tag{3.1}$$

where we assume that the Dirac operator obeys the Ginsparg-Wilson relation [16]

$$\gamma_{d+1} D + \gamma_{d+1} D = D \gamma_{d+1} D. \tag{3.2}$$

The simplest choice is the overlap-Dirac operator [2]

$$D = 1 - A(A^\dagger A)^{-1/2}, \qquad A = 1 - D_{\mathrm{w}}, \tag{3.3}$$

where $D_{\mathrm{w}}$ is the standard Wilson-Dirac operator

$$D_{\mathrm{w}} = \frac{1}{2}[\gamma_\mu(\nabla_\mu^* + \nabla_\mu) - \nabla_\mu^* \nabla_\mu]. \tag{3.4}$$

The covariant derivative $\nabla_\mu$ in the reduced model is defined by eq. (2.10) and $\nabla_\mu^* = \psi - U_\mu^\dagger \psi$. For the overlap-Dirac operator to be well-defined, we require that the gauge field is admissible [26, 27, 1]

$$\|1 - U_{\mu\nu}\| = \|1 - U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\| < \epsilon, \tag{3.5}$$

[5] Note that $[T_\mu^{0\dagger}, T_\nu^0] = 0$.

where $\epsilon$ is a certain constant.

We make a change of variables in eq. (3.1), $\psi \to \psi + \delta\psi$ and $\overline{\psi} \to \overline{\psi} + \delta\overline{\psi}$, where [29]

$$\delta\psi = i\gamma_{d+1}\Big(1 - \frac{1}{2}D\Big)\psi, \qquad \delta\overline{\psi} = i\overline{\psi}\Big(1 - \frac{1}{2}D\Big)\gamma_{d+1}. \tag{3.6}$$

The fermion action does not change under this substitution due to the Ginsparg-Wilson relation. The fermion measure however gives rise to a non-trivial jacobian $Q$ and we have

$$\langle \delta\mathcal{O} \rangle_{\mathrm{F}} = 2iQ\langle \mathcal{O} \rangle_{\mathrm{F}}, \qquad Q = \mathrm{tr}\,\gamma_{d+1}\Big(1 - \frac{1}{2}D\Big). \tag{3.7}$$

We regard this jacobian as "axial anomaly" in the reduced model, because if it were not present, a naive Ward-Takahashi identity $\langle \delta\mathcal{O} \rangle_{\mathrm{F}} = 0$ would be concluded from the symmetry of the fermion action.

It is well-known that the combination $Q$ is an integer [13, 30]. To see this, one notes that the hermitian matrix $\gamma_{d+1}D$ and $\gamma_{d+1}(1-D/2)$ anti-commute to each other as a consequence of the Ginsparg-Wilson relation. If one evaluates the trace in $Q$ by using eigenfunctions of $\gamma_{d+1}D$, therefore, only zero-modes of $\gamma_{d+1}D$ contribute; $Q$ is given by a sum of $\gamma_{d+1}$ eigenvalues of zero-modes, i.e, the index. One may thus regard $Q$ as the topological charge in the reduced model [1].

In general, it is not easy to write down $Q$ directly in terms of the reduced gauge field $U_\mu$. Nevertheless, at least for special configurations such that

$$U_\mu = \Omega u_\mu T_\mu^0 \Omega^\dagger, \tag{3.8}$$

we can find the explicit form of $Q$ in terms of $U_\mu$ by using the correspondence to a U(1) lattice gauge theory in the previous section. We first note that the unitary matrix $\Omega$ does not contribute to $Q$, because $Q$ is gauge invariant and $\Omega$ is the gauge transformation in the reduced model. Then the gauge field has the form (2.11). According to the argument in the previous section, the system is completely identical to a U(1) gauge theory. In particular, the trace in eq. (3.7) is replaced by the sum over all lattice sites. So we have

$$Q = \sum_{x\in\Gamma} \mathrm{tr}\,\gamma_{d+1}\Big[1 - \frac{1}{2}D(x,x)\Big], \tag{3.9}$$

where the U(1) gauge field is given by the diagonal elements of the matrix $u_\mu$. Note that the admissibility (3.5) is promoted to the admissibility in the U(1) theory, because $\|1 - u_{\mu\nu}(x)\| < \epsilon$ for all $x$ from eq. (2.17) (recall that $U_{\mu\nu}$ is a diagonal matrix).

Under the admissibility, a simple expression of $Q$ (3.9) in terms of the U(1) gauge field is known. It is [18]

$$\begin{aligned} Q = \frac{(-1)^{d/2}}{(4\pi)^{d/2}(d/2)!} \sum_{x\in\Gamma} & \epsilon_{\mu_1\nu_1\cdots\mu_{d/2}\nu_{d/2}} f_{\mu_1\nu_1}(x) f_{\mu_2\nu_2}(x+\hat{\mu}_1+\hat{\nu}_1)\cdots \\ & \times f_{\mu_{d/2}\nu_{d/2}}(x+\hat{\mu}_1+\hat{\nu}_1+\cdots+\hat{\mu}_{d/2-1}+\hat{\nu}_{d/2-1}), \end{aligned} \tag{3.10}$$

where the U(1) field strength is defined by[6]

$$f_{\mu\nu}(x) = \frac{1}{i}\ln u_{\mu\nu}(x), \qquad -\pi < f_{\mu\nu}(x) \le \pi. \tag{3.11}$$

Thus, we immediately find, in the reduced model

$$Q = \frac{i^{d/2}}{(4\pi)^{d/2}(d/2)!}\,\epsilon_{\mu_1\nu_1\cdots\mu_{d/2}\nu_{d/2}}\,\mathrm{tr}(\ln U_{\mu_1\nu_1})T^0_{\mu_1}T^0_{\nu_1}(\ln U_{\mu_2\nu_2})T^{0\dagger}_{\nu_1}T^{0\dagger}_{\mu_1}\cdots$$
$$\times T^0_{\mu_1}T^0_{\nu_1}\cdots T^0_{\mu_{d/2-1}}T^0_{\nu_{d/2-1}}(\ln U_{\mu_{d/2}\nu_{d/2}})T^{0\dagger}_{\nu_{d/2-1}}T^{0\dagger}_{\mu_{d/2-1}}\cdots T^{0\dagger}_{\nu_1}T^{0\dagger}_{\mu_1}. \tag{3.12}$$

Note that $T^0_{\mu_1}T^0_{\nu_1}(\ln U_{\mu_2\nu_2})T^{0\dagger}_{\nu_1}T^{0\dagger}_{\mu_1}$ for example is Lie-algebra valued. Since this is a diagonal matrix, it belongs to the Cartan sub-algebra. Therefore, $Q$ is given by a linear combination of $\mathrm{str}(T^{a_1}\cdots T^{a_{d/2}})$, where $T^a$ is a (Cartan) generator of the gauge group in the fundamental representation.

We want to evaluate $Q$ for admissible configurations. Fortunately, admissible U(1) gauge fields have been completely classified by Lüscher [25]. The most general form of the U(1) link variable such that $\|1 - u_{\mu\nu}(x)\| < \epsilon$ for all $x$ is given by[7]

$$u_\mu(x) = \omega(x)v^{[m]}_\mu(x)u^{[w]}_\mu(x)e^{ia^{\mathrm{T}}_\mu}(x)\omega(x+\hat\mu)^{-1}. \tag{3.13}$$

In this expression, $\omega(x) \in \mathrm{U}(1)$ is the U(1) gauge transformation. The field $u^{[w]}_\mu(x)$ is defined by

$$u^{[w]}_\mu(x) = \begin{cases} w_\mu, & \text{for } x_\mu = 0,\\ 1, & \text{otherwise,}\end{cases} \qquad w_\mu \in \mathrm{U}(1), \tag{3.14}$$

and it has vanishing field strength $f_{\mu\nu}(x) = 0$ and carries the Wilson (or Polyakov) line, $\prod_{s=0}^{L-1} u^{[w]}_\mu(s\hat\mu) = w_\mu$. The field $v^{[m]}_\mu(x)$ is defined by

$$v^{[m]}_\mu(x) = \exp\Big[-\frac{2\pi i}{L^2}\Big(L\delta_{x_\mu,L-1}\sum_{\nu>\mu} m_{\mu\nu}x_\nu + \sum_{\nu<\mu} m_{\mu\nu}x_\nu\Big)\Big], \tag{3.15}$$

and carries a constant field strength

$$f_{\mu\nu}(x) = \frac{2\pi}{L^2}\,m_{\mu\nu}, \tag{3.16}$$

where the "magnetic flux" $m_{\mu\nu}$ is an integer bounded by[8]

$$|m_{\mu\nu}| < \frac{\epsilon'}{2\pi}\,L^2. \tag{3.17}$$

[6]For the cohomological analysis to apply, $\epsilon$ in eq. (3.5) has to be smaller than 1. Then the logarithm of the plaquette always remains within the principal branch because $|f_{\mu\nu}(x)| < \pi/3$.

[7]When the gauge group is SU($N$), $\prod_{x\in\Gamma} u_\mu(x)$ must be unity. This requires that $w_\mu \in \mathbb{Z}_{L^{d-1}}$ and $\prod_{x\in\Gamma} v^{[m]}_\mu(x) = \exp[-\pi i L^{d-2}(L-1)\sum_\nu m_{\mu\nu}] = 1$. The latter is always satisfied for $d > 2$.

[8]$\epsilon' = 2\arcsin(\epsilon/2)$.

The "transverse" gauge potential $a_\mu^{\mathrm{T}}(x)$ is defined by[9]

$$\begin{aligned}
&\partial_\mu^* a_\mu^{\mathrm{T}}(x)=0, \qquad \sum_{x\in\Gamma} a_\mu^{\mathrm{T}}(x)=0,\\
&|f_{\mu\nu}(x)|=|\partial_\mu a_\nu^{\mathrm{T}}(x)-\partial_\nu a_\mu^{\mathrm{T}}(x)+2\pi m_{\mu\nu}/L^2|<\epsilon'.
\end{aligned} \tag{3.18}$$

Note that the space of $a_\mu^{\mathrm{T}}(x)$ is contractible.

In terms of $N\times N$ matrix in the reduced model, the above admissible configuration is represented by [$\omega(x)$ can be absorbed into $\Omega$ in eq. (3.8)]

$$U_\mu=u_\mu T_\mu^0=v_\mu^{[m]}u_\mu^{[w]}e^{ia_\mu^{\mathrm{T}}}T_\mu^0, \qquad a_\mu^{\mathrm{T}}-T_\mu^{0\dagger}a_\mu^{\mathrm{T}}T_\mu^0=0, \qquad \mathrm{tr}\, a_\mu^{\mathrm{T}}=0, \tag{3.19}$$

where

$$u_\mu^{[w]}=1\otimes\cdots\otimes 1\otimes W_\mu\otimes 1\otimes\cdots\otimes 1, \tag{3.20}$$

with

$$W_\mu=\begin{pmatrix} w_\mu & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix}, \qquad w_\mu\in \mathrm{U}(1), \tag{3.21}$$

and

$$v_\mu^{[m]}=Y^{m_{1\mu}}\otimes\cdots\otimes Y^{m_{\mu-1\mu}}\otimes Z_\mu, \tag{3.22}$$

where

$$Y=\begin{pmatrix} 1 & & & & \\ & \zeta & & & \\ & & \zeta^2 & & \\ & & & \ddots & \\ & & & & \zeta^{(L-1)}\end{pmatrix}, \qquad \zeta=e^{2\pi i/L^2}, \tag{3.23}$$

and

$$\begin{aligned}
Z_\mu=&\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0\end{pmatrix}\otimes 1\otimes\cdots\otimes 1\\
&+\begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1\end{pmatrix}\otimes S^{m_{\mu+1\mu}}\otimes\cdots\otimes S^{m_{d\mu}}.
\end{aligned} \tag{3.24}$$

For the configuration (3.19) or equivalently for eq. (3.13), from eq. (3.10), we have

$$Q=\frac{(-1)^{d/2}}{2^{d/2}(d/2)!}\,\epsilon_{\mu_1\nu_1\cdots\mu_{d/2}\nu_{d/2}}m_{\mu_1\nu_1}m_{\mu_2\nu_2}\cdots m_{\mu_{d/2}\nu_{d/2}}, \tag{3.25}$$

[9] $\partial_\mu$ and $\partial_\mu^*$ denote the forward and the backward difference operators, $\partial_\mu f(x)=f(x+\hat\mu)-f(x)$, $\partial_\mu^* f(x)=f(x)-f(x-\hat\mu)$, respectively.

which is manifestly an integer. This is the general form of the axial anomaly in the reduced model within the U(1) embedding. We note that $|Q| < \epsilon'^{d/2} d! L^d / [(4\pi)^{d/2}(d/2)!] \propto N$.

It is interesting to consider the pure gauge action

$$\begin{aligned} S_G &= N\beta \sum_{\mu,\nu} \mathrm{Re}\,\mathrm{tr}(1 - U_{\mu\nu}) \\ &= N\beta \sum_{\mu,\nu} \sum_{x\in\Gamma} [1 - \cos f_{\mu\nu}(x)], \end{aligned} \tag{3.26}$$

of an admissible configuration[10] with $Q \neq 0$. For $u_\mu(x) = v_\mu^{[m]}(x)$, this reads,

$$\begin{aligned} S_G &= N\beta \sum_{\mu,\nu} \sum_{x\in\Gamma} \left(1 - \cos \frac{2\pi}{L^2} m_{\mu\nu}\right) \\ &\overset{N\to\infty}{\to} 2\pi^2 \beta N^{2-4/d} \sum_{\mu,\nu} m_{\mu\nu}^2, \end{aligned} \tag{3.27}$$

where we have used $N = L^d$. Thus, as noted in ref. [1], the action of $u_\mu(x) = v_\mu^{[m]}(x)$ remains finite only for $d = 2$ (allowed fluctuations of $a_\mu^{\mathrm{T}}(x)$ are of $O(1/N)$). In fact, this behavior persists for general admissible configurations:

$$\begin{aligned} S_G &\geq N\beta \sum_{\mu,\nu} \sum_{x\in\Gamma} \alpha f_{\mu\nu}(x)^2 \\ &= N\beta\alpha \sum_{\mu,\nu} \sum_{x\in\Gamma} \left\{ [\partial_\mu a_\nu^{\mathrm{T}}(x) - \partial_\nu a_\mu^{\mathrm{T}}(x)]^2 + \frac{4\pi^2}{L^4} m_{\mu\nu}^2 \right\} \\ &\geq 4\pi^2 \alpha\beta N^{2-4/d} \sum_{\mu,\nu} m_{\mu\nu}^2, \end{aligned} \tag{3.28}$$

where, in the first line, we have noted $\cos x \leq 1 - \alpha x^2$ for $0 < \alpha < 1/2$. This lower bound for the action shows that the action of a configuration with $Q \neq 0$ always diverges for $N \to \infty$ if $d > 2$, within the U(1) embedding.

# 4. Obstruction to a smooth measure in reduced chiral gauge theories

In this section, we consider a Weyl fermion coupled to the reduced gauge field and show that there is an obstruction to a smooth fermion measure; this might be regarded as a remnant of the gauge anomaly of the original theory.

The average over fermion variables is defined by[11]

$$\langle \mathcal{O} \rangle_{\mathrm{F}} = \int \mathrm{D}[\psi]\mathrm{D}[\overline{\psi}]\, \mathcal{O} \exp(-\overline{\psi} D \psi), \tag{4.1}$$

[10] To make the admissibility and a smoothness of the action compatible, this action might be too simple [25].

[11] The presentation in this section closely follows the framework of refs. [25, 28]. We refer to refs. [25, 28] and references therein for further details.

where Weyl fermions are subject of the chirality constraint

$$\hat{P}_{\mathrm{H}}\psi = \psi, \qquad \overline{\psi}P_{\widetilde{\mathrm{H}}} = \overline{\psi}. \tag{4.2}$$

In this expression, the chiral projectors are defined by $\hat{P}_{\pm} = (1 \pm \hat{\gamma}_{d+1})/2$ and $P_{\pm} = (1 \pm \gamma_{d+1})/2$ and $\hat{\gamma}_{d+1}$ is the modified chiral matrix, $\hat{\gamma}_{d+1} = \gamma_{d+1}(1 - D)$; H denotes the chirality $\mathrm{H} = \pm$ and $\widetilde{\mathrm{H}} = \mp$. Note that the Ginsparg-Wilson relation implies $(\hat{\gamma}_{d+1})^2 = 1$ and $D\hat{\gamma}_{d+1} = -\gamma_{d+1}D$. This definition thus provides a consistent decomposition of the fermion action, $\overline{\psi}D\hat{P}_{\mathrm{H}}\psi = \overline{\psi}P_{\widetilde{\mathrm{H}}}D\psi$.

The fermion integration measure is defined as usual by $\mathrm{D}[\psi] = \prod_j \mathrm{d}c_j$, where $c_j$ is the expansion coefficient in $\psi = \sum_j v_j c_j$ with respect to an orthonormal basis $v_j$ in the constrained space $\hat{P}_{\mathrm{H}}v_j = v_j$ $[(v_k, v_j) = \delta_{kj}]$.[12] However, since the chiral projector $\hat{P}_{\mathrm{H}}$ depends on the gauge field, and the constraint $\hat{P}_{\mathrm{H}}v_j = v_j$ alone does not specify basis vectors uniquely, it is not obvious how one should change the basis vectors $v_j$ when the gauge field is varied. This implies that there exists a gauge-field-depending phase ambiguity in the measure. This problem is formulated as follows:

One can cover the space of admissible configurations by open local coordinate patches $X_A$ labelled by an index $A$. Within each patch, smooth basis vectors $v_j^A$ can always be found, because $\hat{P}_{\mathrm{H}}$ depends smoothly on the gauge field. In the intersection $X_A \cap X_B$, however, two bases are in general different and related by a unitary transformation, $v_j^B = \sum_l v_l^A \tau(A \to B)_{lj}$ and $c_j^B = \sum_l \tau(A \to B)_{jl}^{-1} c_l^A$. The fermion measures defined with respect to each basis are thus related as

$$\mathrm{D}[\psi]^B = g_{AB}\,\mathrm{D}[\psi]^A, \qquad g_{AB} = \det \tau(A \to B) \in \mathrm{U}(1). \tag{4.3}$$

Hence the above setup defines a U(1) fiber bundle over the space of admissible configurations, $g_{AB}$ being the transition function. The smoothness of the fermion integration measure (i.e., single-valued-ness of $\langle \mathcal{O} \rangle_{\mathrm{F}}$) thus requires that this U(1) bundle is trivial and that one can adjust bases $v_j^A$ and $v_j^B$ such that the transition function is unity, $g_{AB} = 1$ on $X_A \cap X_B$.[13] Whether this is the case or not eventually depends on the properties of the chiral projector $\hat{P}_{\mathrm{H}}$ and of the base manifold, the space of admissible configurations.

We consider an infinitesimal variation of the gauge field

$$\delta_\eta U_\mu = \eta_\mu U_\mu, \qquad \eta_\mu = \eta_\mu^a T^a, \tag{4.4}$$

and introduce the "measure term" in the patch $X_A$ by

$$\mathfrak{L}_\eta^A = i \sum_j (v_j^A, \delta_\eta v_j^A), \tag{4.5}$$

[12] For the anti-fermion, $\mathrm{D}[\overline{\psi}] = \prod_k \mathrm{d}\overline{c}_k$, where $\overline{\psi} = \sum_k \overline{c}_k \overline{v}_k$ and $\overline{v}_k P_{\widetilde{\mathrm{H}}} = \overline{v}_k$. Basis vectors $\overline{v}_k$ can be chosen to be independent of the gauge field.

[13] Under a change of bases, the transition function transforms according to $g_{AB} \to h_A g_{AB} h_B^{-1}$ on $X_A \cap X_B$, where $h_A$ $(h_B)$ is a determinant of the transformation matrix in the patch $X_A$ $(X_B)$.

which parameterizes the above phase ambiguity. The measure terms in adjacent two patches are related by

$$\mathfrak{L}_\eta^A = \mathfrak{L}_\eta^B - i\delta_\eta \ln g_{AB}, \qquad \text{on} \quad X_A \cap X_B. \tag{4.6}$$

Thus the measure term is the connection of the U(1) bundle. We may introduce a local coordinate $(t, s, \ldots)$ in $X_A$ and define the U(1) curvature by

$$\partial_t \mathfrak{L}_\sigma^A - \partial_s \mathfrak{L}_\tau^A = i \operatorname{tr}(\hat{P}_{\mathrm{H}}[\partial_t \hat{P}_{\mathrm{H}}, \partial_s \hat{P}_{\mathrm{H}}]), \tag{4.7}$$

where the variation vectors have been defined by

$$\tau_\mu = \partial_t U_\mu U_\mu^\dagger, \qquad \sigma_\mu = \partial_s U_\mu U_\mu^\dagger. \tag{4.8}$$

Equation (4.7), which follows from eq. (4.5) and $[\partial_t, \partial_s] = 0$, shows that the curvature is independent of the referred patch, as it should be the case.[14]

Take a closed 2 dimensional surface $\mathcal{M}$ in the space of admissible configurations. The first Chern number of the above U(1) bundle is then given by

$$\mathcal{I} = \frac{1}{2\pi} \int_{\mathcal{M}} \mathrm{d}t \, \mathrm{d}s \, i \operatorname{tr}(\hat{P}_{\mathrm{H}}[\partial_t \hat{P}_{\mathrm{H}}, \partial_s \hat{P}_{\mathrm{H}}]). \tag{4.9}$$

If this integer does not vanish, $\mathcal{I} \neq 0$, the U(1) bundle is non-trivial and a smooth fermion measure does not exist according to the above argument. If $\mathcal{I} \neq 0$, we may regard this as a remnant of the gauge anomaly, because in the classical continuum limit of the original gauge theory *before the reduction*, $\mathcal{I}$ is proportional to the anomaly $\operatorname{str}[R(T^{a_1}) \cdots R(T^{a_{d/2+1}})]$, where $R$ is the gauge representation of the Weyl fermion [28, 32].[15]

The above is for the reduced model. The correspondence to the U(1) theory in section 2 is applied also to this system of Weyl fermion, because couplings to

[14] The above U(1) bundle, the connection and the curvature were first addressed in ref. [31] in the context of the overlap.

[15] Under the infinitesimal gauge transformation, $\delta_\eta U_\mu = [\omega, U_\mu]$, $\delta_\eta \psi = \omega\psi$ and $\delta_\eta \overline{\psi} = -\overline{\psi}\omega$, one can show that

$$\begin{aligned} &\delta_\eta \langle \mathcal{O} \rangle_{\mathrm{F}} = \langle \delta_\eta \mathcal{O} \rangle_{\mathrm{F}} + i\omega^a [\mathcal{A}^a - (\nabla_\mu^* j_\mu)^a] \langle \mathcal{O} \rangle_{\mathrm{F}}, \\ &\nabla_\mu^* j_\mu = j_\mu - U_\mu^\dagger j_\mu U_\mu, \qquad \mathcal{A}^a = -i \operatorname{tr} T^a \gamma_{d+1} \left(1 - \frac{1}{2} D\right), \end{aligned} \tag{4.10}$$

where $j_\mu$ is the measure current defined by $\mathfrak{L}_\eta = \eta_\mu^a j_\mu^a$, where $\eta_\mu = -\nabla_\mu \omega$ and $\nabla_\mu \omega = U_\mu \omega U_\mu^\dagger - \omega$. The gauge anomaly in this framework is thus given by the combination, $\mathcal{G}^a = \mathcal{A}^a - (\nabla_\mu^* j_\mu)^a$. An evaluation of $\mathcal{G}^a$ is however somewhat subtle because it is ambiguous depending on the measure current which specifies the fermion integration measure. For conventional chiral gauge theories, assuming the locality of the measure current, it is possible to argue that this ambiguity can be absorbed into a gauge variation of a local functional (i.e., a local counter-term). In the reduced model, however, the meaning of the locality of the measure current $j_\mu^a$ is not clear. This is the reason why we study the first Chern number $\mathcal{I}$ instead of the gauge anomaly $\mathcal{G}^a$ itself.

the gauge field, even in the chiral constraint (4.2), arise only through the covariant derivative (2.10). Hence, under the assumption (2.11), the above system is identical to a U(1) chiral gauge theory defined on the lattice $\Gamma$ in which the Ginsparg-Wilson Dirac operator is employed. In terms of the U(1) lattice theory, the first Chern number reads

$$\mathcal{I} = \frac{1}{2\pi}\int_{\mathcal{M}} \mathrm{d}t\,\mathrm{d}s\, i\sum_{x\in\Gamma} \mathrm{tr}(\hat{P}_{\mathrm{H}}[\partial_t\hat{P}_{\mathrm{H}}, \partial_s\hat{P}_{\mathrm{H}}])(x,x). \tag{4.11}$$

We will evaluate $\mathcal{I}$ in this U(1) picture. Since this $\mathcal{I}$ is an integer, it is invariant under a smooth deformation of admissible configurations defined on $\mathcal{M}$. This implies that $\mathcal{I}$ is independent of the transverse potential $a^{\mathrm{T}}_\mu(x)$ in eq. (3.13), because these degrees of freedom can be deformed to the trivial value, $a^{\mathrm{T}}_\mu(z)\to 0$, without affecting the admissibility.

To evaluate $\mathcal{I}$ in the picture of U(1) lattice theory, it is convenient to introduce Lüscher's topological field in $d+2$-dimensional space [28]. To define this field, we introduce continuous two dimensional space whose coordinates are $t$ and $s$. The U(1) gauge field is assumed to depend also on these additional coordinates, $u_\mu(z)$ where $z=(x,t,s)$. We further introduce gauge potentials $a_t(z)$, $a_s(z)\in\mathfrak{u}(1)$ along the continuous directions. The associated field tensor is defined by

$$f_{ts}(z) = \partial_t a_s(z) - \partial_s a_t(z), \tag{4.12}$$

and the covariant derivatives is defined by ($r=t$ or $s$)

$$D^a_r u_\mu(z) = \partial_r u_\mu(z) + i a_r(z) u_\mu(z) - i u_\mu(z) a_r(z+\hat{\mu}). \tag{4.13}$$

For a gauge covariant quantity such that $\hat{P}_{\mathrm{H}}$, it reads

$$D^a_r \hat{P}_{\mathrm{H}} = \partial_r \hat{P}_{\mathrm{H}} + i[a_r, \hat{P}_{\mathrm{H}}]. \tag{4.14}$$

Lüscher's topological field is then defined by[16]

$$q(z) = i\epsilon_{\mathrm{H}}\,\mathrm{tr}\Big\{\Big[\frac{1}{4}\hat{\gamma}_{d+1}[D^a_t\hat{P}_{\mathrm{H}}, D^a_s\hat{P}_{\mathrm{H}}] + \frac{1}{4}[D^a_t\hat{P}_{\mathrm{H}}, D^a_s\hat{P}_{\mathrm{H}}]\hat{\gamma}_{d+1} + \frac{i}{2} f_{ts}\hat{\gamma}_{d+1}\Big](x,x)\Big\}, \tag{4.15}$$

which is a gauge invariant (in $d+2$-dimensional sense) pseudoscalar local field. It can be verified that [28]

$$\sum_{x\in\Gamma} q(z) = i\sum_{x\in\Gamma}\mathrm{tr}\Big[\hat{P}_{\mathrm{H}}[\partial_t\hat{P}_{\mathrm{H}}, \partial_s\hat{P}_{\mathrm{H}}] + \frac{i}{2}\epsilon_{\mathrm{H}}\partial_t(a_s\hat{\gamma}_{d+1}) - \frac{i}{2}\epsilon_{\mathrm{H}}\partial_s(a_t\hat{\gamma}_{d+1})\Big](x,x). \tag{4.16}$$

Thus it is a topological field satisfying

$$\int \mathrm{d}t\,\mathrm{d}s \sum_{x\in\Gamma}\delta q(z) = 0, \tag{4.17}$$

[16] $\epsilon_\pm = \pm 1$.

for any local variation of the gauge fields, $u_\mu(z)$ and $a_r(z)$. Equation (4.16) also shows that

$$\mathcal{I}=\frac{1}{2\pi}\int_{\mathcal{M}}\mathrm{d}t\,\mathrm{d}s\sum_{x\in\Gamma}q(z), \tag{4.18}$$

if the gauge fields, $u_\mu(z)$ and $a_r(z)$, are single-valued on $\mathcal{M}$.

A cohomological analysis again provides an important information on $q(z)$. Using the gauge invariance, the topological property and the pseudoscalar nature of $q(z)$, a cohomological analysis along the line of ref. [22] shows that

$$q^\infty(z)=p(z)+\partial_\mu^* k_\mu^\infty(z)+\partial_t k_s^\infty(z)-\partial_s k_t^\infty(z), \tag{4.19}$$

when the lattice-size is *infinite*, $L\to\infty$. In this expression, $k_\mu^\infty(z)$, $k_t^\infty(z)$ and $k_s^\infty(z)$ are gauge invariant local currents (which is translational invariant) and the main part $p(z)$ of $q^\infty(z)$ is given by[17]

$$\begin{aligned} p(z)=\frac{(-1)^{d/2+1}\epsilon_{\mathrm{H}}}{2(4\pi)^{d/2}(d/2+1)!}\,&\epsilon_{M_1N_1\cdots M_{d/2+1}N_{d/2+1}}f_{M_1N_1}(z)f_{M_2N_2}(z+\hat{M}_1+\hat{N}_1)\cdots\\ &\times f_{M_{d/2+1}N_{d/2+1}}(z+\hat{M}_1+\hat{N}_1+\cdots+\hat{M}_{d/2}+\hat{N}_{d/2}), \end{aligned} \tag{4.20}$$

where $M=(\mu,t,s)$ etc. and we take $\hat{t}=\hat{s}=0$; $f_{r\mu}(z)=u_\mu(z)^{-1}\partial_r u_\mu(z)/i-\partial_\mu a_r(z)$. When $p(z)$ does not depend on $a_r(z)$,[18] one may rewrite $p(z)$ in terms of the reduced gauge field $U_\mu$ in an analogous form as eq. (3.12). Note that $f_{r\mu}(z)$ is given by $(T_\mu^0 U_\mu^\dagger\partial_r U_\mu T_\mu^{0\dagger})_{m(x)m(x)}/i$. Then, by the same way as for eq. (3.12), one sees that $p(z)$ is a linear combination of $\mathrm{str}(T^{a_1}\cdots T^{a_{d/2+1}})$.

Now let us evaluate the first Chern number $\mathcal{I}$ (4.11) by taking a 2 torus $T^2$ as the two-dimensional surface $\mathcal{M}$. We parameterize $T^2$ by $0\le t\le 2\pi$ and $0\le s\le 2\pi$. As already noted, $\mathcal{I}$ is independent of $a_\mu^{\mathrm{T}}(x)$ in eq. (3.13); we can set $a_\mu^{\mathrm{T}}(z)=0$ without loss of generality. Similarly, we may assume that the gauge degrees of freedom $\omega(x)$ and the Wilson-line degrees of freedom $u_\mu^{[w]}(x)$ in eq. (3.13) have the following standard forms:

$$\omega(z)=\exp[iL^t(x)t+iL^s(x)s], \tag{4.21}$$

and

$$u_\mu^{[w]}(z)=\begin{cases}\exp(iJ_\mu^t t+iJ_\mu^s s), & \text{for } x_\mu=0,\\ 1, & \text{otherwise,}\end{cases} \tag{4.22}$$

where $L^r(x)$ and $J_\mu^r$ are integer winding numbers, $L^r(x)$, $J_\mu^r\in\mathbb{Z}$, because these are representatives of the homotopy class of mappings from $T^2$ to $\mathrm{U}(1)=S^1$; any

[17] The numerical coefficient of this expression cannot be determined by the cohomological analysis. We have used a matching with a result in the classical continuum limit [28, 32]; see also ref. [18] and references therein.

[18] For example, when $a_r(z)$ is pure-gauge $a_r(z)=\omega(z)\partial_r\omega(z)^{-1}/i$, a dependence of $p(z)$ on $a_r(z)$ disappears combined with the gauge degrees of freedom $\omega(z)$ in eq. (3.13). This is precisely the situation we will consider below.

mapping can smoothly be deformed into these standard forms without changing the integer $\mathcal{I}$ (4.11).[19] For gauge fields along the continuous directions, we take the pure gauge configuration, $a_r(z) = \omega(z)\partial_r\omega(z)^{-1}/i = -L^r(x)$. Note that this $a_r(z)$ is single-valued on $T^2$ and thus eq. (4.18) holds. Under these restrictions on the gauge fields, we note

$$D_r^a u_\mu(z) u_\mu(z)^{-1} = i J_\mu^r \delta_{x_\mu,0}, \tag{4.23}$$

and

$$f_{\mu\nu}(z) = \frac{2\pi}{L^2} m_{\mu\nu}, \qquad f_{r\mu}(z) = J_\mu^r \delta_{x_\mu,0}, \qquad f_{ts}(z) = 0. \tag{4.24}$$

For the admissible configuration (3.13) with the above restrictions on the gauge fields, it is immediate to evaluate the integral of $q^\infty(z)$:

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}t \int_0^{2\pi} \mathrm{d}s \sum_{x\in\Gamma} q^\infty(z) &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}t \int_0^{2\pi} \mathrm{d}s \sum_{x\in\Gamma} p(z) \\ &= \frac{(-1)^{d/2}\epsilon_{\mathrm{H}}}{2^{d/2-1}(d/2-1)!} \epsilon_{\mu_1\nu_1\cdots\mu_{d/2}\nu_{d/2}} m_{\mu_1\nu_1} \cdots m_{\mu_{d/2-1}\nu_{d/2-1}} J^t_{\mu_{d/2}} J^s_{\nu_{d/2}}, \end{aligned} \tag{4.25}$$

which is an integer. The field $q^\infty(z)$, which is originally defined on the infinite lattice, depends on the gauge-field background defined on the infinite lattice. As this gauge-field configuration on the infinite lattice, we take periodic copies of a gauge-field configuration defined on $\Gamma$. Then, due to the translational invariance, $k_\mu^\infty(z)$ is periodic on $\Gamma$ and we have the first equality. The second equality follows from eq. (4.24).

We can in fact show that (appendix A), using the locality of the Dirac operator, integral (4.25) coincides with eq. (4.18) when the lattice size $L$ is sufficiently large, i.e., when $N$ is sufficiently large. Thus we have

$$\mathcal{I} = \frac{(-1)^{d/2}\epsilon_{\mathrm{H}}}{2^{d/2-1}(d/2-1)!} \epsilon_{\mu_1\nu_1\cdots\mu_{d/2}\nu_{d/2}} m_{\mu_1\nu_1} \cdots m_{\mu_{d/2-1}\nu_{d/2-1}} J^t_{\mu_{d/2}} J^s_{\nu_{d/2}}. \tag{4.26}$$

This shows that $\mathcal{I} \neq 0$ for certain configurations defined on $\Gamma \times T^2$ and there exists an obstruction to a smooth measure on a 2 torus embedded in the space of admissible configurations. As shown in section 3, however, the pure-gauge action of any configuration which leads to $\mathcal{I} \neq 0$ for $\mathcal{M} = T^2$ diverges as $N \to \infty$ when $d > 2$, within the U(1) embedding.

We want to comment on the difference of our result from Neuberger's work [31]. In ref. [31], a torus in the *orbit space*, $\mathfrak{U}/\mathfrak{G}$ where $\mathfrak{U}$ is a connected component of the space of admissible configurations and $\mathfrak{G}$ is the group of gauge transformations, is considered. It was then shown that, when the gauge anomaly is not canceled, $\mathcal{I} \neq 0$ for appropriate configurations. This is an obstruction to define a smooth $\mathfrak{G}$-invariant

[19] When the gauge group is SU($N$), $\omega_\mu \in \mathbb{Z}_{L^{d-1}}$ and a non-trivial winding of the Wilson line is impossible. This leads to, as we will see, $\mathcal{I} = 0$ for $\mathcal{M} = T^2$.

fermion measure, i.e., an obstruction to the gauge invariance. See also refs. [32, 33]. On the other hand, we have shown here that there exists an obstruction to a smooth fermion measure irrespective of its gauge invariance. Even one sacrifices the gauge invariance, there remains an obstruction.

One might argue that if the gauge invariance is sacrificed, there exists at least one possible choice of a smooth fermion measure, the Wigner-Brillouin phase choice [13]. However, there is a simple example with which the Wigner-Brillouin phase choice becomes singular, at least with a use of the overlap Dirac operator (appendix B). So this choice does not provide a counter-example for our result.

## 5. Conclusion

In this paper, we systematically investigated possible chiral anomalies in the reduced model within a framework of the U(1) embedding. When the overlap-Dirac operator is employed for the fermion sector, the gauge-field configuration must be admissible. This admissibility divides the otherwise connected space of gauge-field configurations into many components. Using the classification of ref. [25], we gave a general form of the the axial anomaly $Q$ within the U(1) embedding. We have also shown that there may exist an obstruction to a smooth fermion integration measure in reduced chiral gauge theories, by evaluating the first Chern number $\mathcal{I}$ of a U(1) bundle associated to the fermion measure. In both cases, the pure gauge action of gauge-field configurations which cause these non-trivial phenomena turns to diverge in the 't Hooft $N \to \infty$ limit when $d > 2$. This might imply that the above phenomena are irrelevant in the 't Hooft $N \to \infty$ limit, in which the reduced model is considered to be equivalent to the original gauge theory.

The most important question we did not answer in this paper is an effect of the U(1) embedding to other gauge representations. This is related to a question of the gauge anomaly cancellation in reduced chiral gauge theories. We expect that if the fermion multiplet is anomaly-free in the conventional sense, then the obstruction we found in the reduced model will disappear. To see this, however, we have to evaluate $\mathcal{I}$ for a Weyl fermion belonging to a representation $R$, with the gauge-field configuration[20]

$$R(u_\mu T^0_\mu). \tag{5.1}$$

Of course, it may be possible to imitate the U(1) embedding in other representations by restricting gauge-field configurations as

$$R(U_\mu) = u'_\mu T^{0\prime}_\mu, \tag{5.2}$$

[20]For the "trivial" anomaly-free cases which consist of equal number of right-handed and left-handed Weyl fermions in the fundamental representation, the obstruction $\mathcal{I}$ vanishes because $\mathcal{I}$ is proportional to the chirality $\epsilon_{\rm H}$.

where $R$ is a $N' \times N'$ representation matrix and the shift operator $T^{0\prime}_{\mu}$ is for a lattice with the size $L'$ and $L'^{d} = N'$. A similar argument as this paper will then be applied with this type of embedding. Generally, however, the backgrounds (5.1) and (5.2) do not coincide. For the case of the adjoint representation, a connection of the reduced model to non-commutative lattice gauge theory [34, 35] might be helpful.

Another interesting extension is to embed a lattice gauge theory with a larger gauge group, say SU(2), in the reduced model. This is easily done at least for the fundamental representation by identifying two or more columns of the representation vector as a single lattice site. A freedom of internal space then emerges. With this embedding, we have to analyze the axial anomaly in non-abelian lattice gauge theories defined on a finite-size lattice. As for the corresponding axial anomaly $Q$, there is a conjecture [18], which holds to all orders in perturbation theory, that $Q$ coincides with the Lüscher's topological charge [36]. So, accepting this conjecture, the SU(2) instanton configuration on the lattice [37] with this embedding will provide an example of $Q \neq 0$.

Another direction is to investigate the Witten anomaly [38] in the present setup following the line of argument in refs. [39, 40].

So, there are many things to do with this embedding trick in the reduced model, when a Ginsparg-Wilson type Dirac operator is employed. We hope to come back some of above problems in the near future.

The authors would like to thank Jun Nishimura for valuable discussions. We would like to thank David Adams for pointing out a misleading statement in the first version of this paper. H.S. would like to thank Kiyoshi Okuyama and Kazuya Shimada for helpful discussions on the reduced model. This work is supported in part by Grant-in-Aid for Scientific Research, #12640262, #14046207 (Y.K.) and #13740142 (H.S.).

## A. Proof of eq. (4.26)[21]

When the size of $\Gamma$ becomes infinity, $L \to \infty$, a Ginsparg-Wilson Dirac operator $D(x,y)$ is promoted to a Dirac operator on the infinite lattice $D(x,y) \to D^{\infty}(x,y)$. We assume that these two operators are related by the reflection [25]

$$D(x,y) = \sum_{n \in \mathbb{Z}^d} D^{\infty}(x, y + Ln), \tag{A.1}$$

where the gauge field configuration in the right hand side is given by periodic copies of $\Gamma$ extended to the infinite lattice. This relation actually holds for the overlap-Dirac

[21] A part of this proof was obtained through H.S.'s discussion with Takanori Fujiwara and Keiichi Nagao.

operator. Equation (A.1) implies, when $a_r(z)$ is pure-gauge,

$$
\begin{aligned}
\sum_{x\in\Gamma} q(z) &= i\sum_{x\in\Gamma} \mathrm{tr}(\hat{P}_{\mathrm{H}}[D_t^a\hat{P}_{\mathrm{H}}, D_s^a\hat{P}_{\mathrm{H}}])(x,x)\\
&= i\sum_{n\in\mathbb{Z}^d}\sum_{x\in\Gamma}\sum_{y,z\in\mathbb{Z}^d} \mathrm{tr}\{\hat{P}_{\mathrm{H}}^{\infty}(x,y)[D_t^a\hat{P}_{\mathrm{H}}^{\infty}(y,z)D_s^a\hat{P}_{\mathrm{H}}^{\infty}(z,x+Ln)-(t\leftrightarrow s)]\},
\end{aligned}
\tag{A.2}
$$

where the kernel $\hat{P}_{\mathrm{H}}^{\infty}(x,y)$ is defined from $D^{\infty}(x,y)$. Note that a sum of $q^{\infty}(z)$ over $\Gamma$ in eq. (4.25), $\sum_{x\in\Gamma} q^{\infty}(z)$, coincides with the $n=0$ term of eq. (A.2). On the other hand, from the locality of the Dirac operator (see ref. [25]), it is possible to show bounds

$$
\begin{aligned}
&\|\hat{P}_{\mathrm{H}}^{\infty}(x,y)\| \le \kappa_1(1+\|x-y\|^{\nu_1})e^{-\|x-y\|/\varrho},\\
&\|D_r^a\hat{P}_{\mathrm{H}}^{\infty}(x,y)\| \le \kappa_2(1+\|x-y\|^{\nu_2})e^{-\|x-y\|/\varrho}\max_{x,\mu}|D_r^a u_\mu(z)u_\mu(z)^{-1}|,
\end{aligned}
\tag{A.3}
$$

where the constants $\kappa_1$, $\kappa_2$, $\nu_1$ and $\nu_2$ are independent of the gauge field. We thus have the bound

$$
\left|\sum_{x\in\Gamma} q(z)-\sum_{x\in\Gamma} q^{\infty}(z)\right| \le \kappa_3 L^{\nu_3}e^{-L/\varrho}\max_{\mu}|J_\mu^t|\max_{\nu}|J_\nu^s|,
\tag{A.4}
$$

where a use of eq. (4.23) has been made. This shows

$$
\left|\mathcal{I}-\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}t\int_0^{2\pi}\mathrm{d}s\sum_{x\in\Gamma} q^{\infty}(z)\right| \le \kappa_4 L^{\nu_4}e^{-L/\varrho}\max_{\mu}|J_\mu^t|\max_{\nu}|J_\nu^s|,
\tag{A.5}
$$

and, when the lattice size is sufficiently large, say $L/\varrho > n$, the integer $\mathcal{I}$ and the integer (4.25) coincide. The required lattice-size for this coincidence however may depend on the gauge-field configuration through the winding numbers $J_\mu^r$.

## B. Wigner-Brillouin phase choice may become singular[22]

Consider a one-parameter family of gauge-field configurations in U(1) theory:

$$
u_\mu^{(\tau)}(x) = \begin{cases} e^{i\pi\tau}, & \text{for } \mu=1,\\ 1, & \text{otherwise,}\end{cases}
\tag{B.1}
$$

where $0\le\tau\le 1$. The field strength of these configurations vanishes, $f_{\mu\nu}^{(\tau)}(x)=0$, so these are admissible configurations. The modified chiral matrix and the projection operator corresponding to these configurations will be denoted by $\hat{\gamma}_{d+1}^{(\tau)}$ and $\hat{P}_{\mathrm{H}}^{(\tau)}$. From the definition of the overlap-Dirac operator, one then finds

$$
\hat{\gamma}_{d+1}^{(\tau)}\psi = \gamma_{d+1}(-i\gamma_1\sin\pi\tau+\cos\pi\tau)\psi,
\tag{B.2}
$$

[22] The following example was suggested to us by Martin Lüscher in the context of general lattice chiral gauge theories.

for any *constant* spinor $\psi$. This implies

$$\hat{P}_{\mathrm{H}}^{(1)}\psi = \hat{P}_{\widetilde{H}}^{(0)}\psi. \tag{B.3}$$

Now, in the Wigner-Brillouin phase choice, the phase ambiguity of the fermion measure is fixed by imposing $\det(v_j^{(0)}, v_k^{(\tau)})$ be real positive, where basis vectors satisfy $\hat{P}_{\mathrm{H}}^{(0)}v_j^{(0)} = v_j^{(0)}$ and $\hat{P}_{\mathrm{H}}^{(\tau)}v_j^{(\tau)} = v_j^{(\tau)}$. This determinant, however, vanishes at $\tau = 1$ because $\hat{P}_{H}^{(1)}\psi$ is contained in $v_j^{(1)}$ and

$$(v_j^{(0)}, \hat{P}_{\mathrm{H}}^{(1)}\psi) = (v_j^{(0)}, \hat{P}_{\widetilde{\mathrm{H}}}^{(0)}\psi) = 0. \tag{B.4}$$

Therefore the Wigner-Brillouin phase choice becomes singular at $\tau = 1$.

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